

THE KRULL AND GLOBAL DIMENSION OF THE TENSOR PRODUCT OF n -DIMENSIONAL QUANTUM TORI

1. INTRODUCTION

Let F be a field. The associative F -algebra generated by X_1, \dots, X_n together with their inverses satisfying the relations

$$(1) \quad X_i X_j = \lambda_{ij} X_j X_i \quad \lambda_{ij} \in F \setminus \{0\} \quad \forall i, j \in \{1, \dots, n\}$$

is known as the quantum torus as it is a quantum deformation of the coordinate algebra of the torus. It plays an important role in non-commutative geometry, quantum groups and group representation theory. It is also interesting in its own right being an example of a noncommutative polynomial algebra.

The case $n = 2$ is of special interest. Here the defining relation is $XY = qYX$ where q is a nonzero scalar in F and we denote the resulting algebra as B_q . It was shown in [J] and [L] that when q is not a root of unity, B_q resembles the first Weyl algebra $A_1(k)$ where k is a field of characteristic zero. In [J] iterated tensor products of the type

$$B_{q_1} \otimes_F \cdots \otimes_F B_{q_k}$$

were also considered.

Returning to the general case we note that the *multiparameters* λ_{ij} satisfy the conditions

$$\lambda_{ii} = 1 = \lambda_{ij} \lambda_{ji} \quad \forall i, j \in \{1, \dots, n\}.$$

Let $\Lambda := (\lambda_{ij})$. We denote the quantum torus defined in (1) by $P(\Lambda)$. We note that the algebras $P(\Lambda)$ are precisely the twisted group algebras $F * A$ of finitely generated free abelian groups A over F . In [MP], J.C. McConnell and J.J. Pettit investigated the ring theoretic dimensions of the algebras $P(\Lambda)$. It was shown in this paper that the Krull and global dimensions of these algebras coincide and if d denotes this common value then $1 \leq d \leq n$. Henceforth by the dimension of $P(\Lambda)$ we shall mean any of these dimensions.

A criterion ([MP, Corollary 3.8]) for the exact value of d in terms of certain partial localizations of $P(\Lambda)$ was also given in the same paper. It was moreover conjectured that if $F * A$ is the underlying twisted group algebra of $P(\Lambda)$, then d is the supremum of the ranks of the subgroups $B \leq A$ for which the subalgebra $F * B$ is commutative. Using the criterion for d mentioned above and a geometric invariant for $F * A$ -modules introduced and studied in [BG1]–[BG3], this conjecture was shown to be true by C.J.B. Brookes in [B].

Given quantum tori $F * A_1$ and $F * A_2$ we can take their tensor product over F (this turns out to be a twisted group algebra $F * (A_1 \times A_2)$). The question then arises as to how the dimension of this tensor product is related to the dimensions of the (tensor) factors $F * A_1$ and $F * A_2$.

From the basic properties of tensor product of algebras and the result of Brookes it immediately follows that

$$(2) \quad \dim(F * A_1 \otimes_F F * A_2) \geq \dim(F * A_1) + \dim(F * A_2),$$

where we denote by $\dim(F * A)$ either the Krull or the global dimension of the algebra $F * A$. In other words the dimension is in general *super-additive* with respect to tensoring.

It is easy to find examples where equality does not hold in (2). As the tensor product $F * A_1 \otimes_F F * A_2$ is a twisted group algebra $F * (A_1 \times A_2)$, its dimension can be at most $\text{rk}(A_1) + \text{rk}(A_2)$.

Our first result gives the following upper bound for the dimension of the tensor product assuming that the dimension of each algebra $F * A_i$ is not the maximum possible.

Theorem 5.7. *Given algebras $F * A_1$ and $F * A_2$ suppose that $\dim(F * A_i) < \text{rk}(A_i)$ for $i \in \{1, 2\}$. Let $d := \dim(F * A_1 \otimes_F F * A_2)$. Then*

$$(3) \quad d \leq \min\{\dim(F * A_1) + \text{rk}(A_2), \dim(F * A_2) + \text{rk}(A_1)\} - 1.$$

It may be of interest to know when is the dimension of a tensor product *additive* with respect to tensoring. In this connection, the following corollary can be deduced from Theorem 5.7.

Corollary 5.11. *Given twisted group algebras $F * A_1$ and $F * A_2$, if $\dim(F * A_i) = \text{rk}(A_i) - 1$ for some $i \in \{1, 2\}$ then*

$$\dim(F * A_1 \otimes_F F * A_2) = \dim(F * A_1) + \dim(F * A_2)$$

The upper bound of (3) is the best possible. For example, let $\Lambda = (\lambda_{ij})$ be a multiplicatively antisymmetric $n \times n$ matrix such that the subset $\{\lambda_{ij} \mid 1 \leq i < j \leq n\}$ of F is multiplicatively independent. By [MP, Corollary 3.10], $\dim(P_\Lambda) = 1$. Let Λ' be the transpose of Λ . Then Λ' is also multiplicatively antisymmetric and $\dim(P_{\Lambda'}) = 1$ by the same result.

Suppose that $P_{\Lambda'}$ is generated over F by the indeterminates X'_i , ($i \in \{1, \dots, n\}$) together with their inverses. Our choice of the defining multiparameters ensures that the monomials

$$\{X_i \otimes X'_i \mid 1 \leq i \leq n\}$$

commute mutually in $P_\Lambda \otimes_F P_{\Lambda'}$. But then $\dim(P_\Lambda \otimes_F P_{\Lambda'}) \geq n$ in view of Brookes' theorem (Theorem 5.1).

Writing P_Λ as $F * A$ and similarly $P_{\Lambda'}$ as A' , where both A and A' have rank n , we have in view of (3) that

$$\begin{aligned} \dim(P_\Lambda \otimes_F P_{\Lambda'}) &\leq \min\{\dim(F * A) + \text{rk}(A'), \dim(F * A') + \text{rk}(A)\} - 1 \\ &= \min\{1 + n, 1 + n\} - 1 \\ &= n. \end{aligned}$$

This shows that the upper bound in (3) is the best possible. There are also cases in which the upper bound in (3) is not attained by the dimension of the tensor product $F * A_1 \otimes_F F * A_2$. Theorem 5.14 gives sufficient conditions for this.

Theorem 5.14. *Let $F * A_1$ and $F * A_2$ be twisted group algebras such that*

- (i) $\dim(F * A_i) \geq 2$,
- (ii) $\text{rk}(A_i) - \dim(F * A_i) \geq 2$,
- (iii) $F * A_i$ has center F .

Let $d := \dim(F * A_1 \otimes F * A_2)$. Then

$$d < \min\{\dim(F * A_1) + \text{rk}(A_2), \dim(F * A_2) + \text{rk}(A_1)\} - 1.$$

It is convenient to define codimension of the algebra $F * A$ as

$$\text{co-dim}(F * A) = \text{rk}(A) - \dim(F * A).$$

We can deduce the following corollary from Theorem 5.14 which gives further cases where the dimension is additive with respect to tensor products.

Corollary 5.15. *Suppose that the algebras $F * A_1$ and $F * A_2$ satisfy the following conditions*

- (i) $\dim(F * A_1), \dim(F * A_2) \geq 2$,
- (ii) $\text{co-dim}(F * A_1) \geq \text{co-dim}(F * A_2) = 2$,
- (iii) $F * A_i$ has center F .

Then

$$\dim(F * A_1 \otimes F * A_2) = \dim(F * A_1) + \dim(F * A_2)$$

This article is organised as follows. We begin by reviewing twisted group algebras $F * A$ and their localizations in Section 2. In Section 3 some facts on $F * A$ -modules are recalled. Section 4 gives an exposition of a geometric invariant of Brookes and Groves for $F * A$ -modules which we use in our investigations. The problem of the dimension of tensor products is discussed in Section 5.

2. TWISTED GROUP ALGEBRAS AND CROSSED PRODUCTS

Let $F^* := F \setminus \{0\}$. Let A denote a finitely generated free abelian group. We denote by $\text{rk}(A)$ the rank of A . An F -algebra \mathcal{A} is a *twisted group algebra* $F * A$ of A over F if \mathcal{A} has a copy $\overline{A} := \{\bar{a} : a \in A\}$ of A which is an F -basis so that the multiplication in \mathcal{A} satisfies

$$(4) \quad \bar{a}_1 \bar{a}_2 = \tau(a_1, a_2) \overline{a_1 a_2} \quad \forall a_1, a_2 \in A,$$

where $\tau : A \times A \rightarrow F^*$ is a function satisfying

$$\tau(a_1, a_2) \tau(a_1 a_2, a_3) = \tau(a_2, a_3) \tau(a_1, a_2 a_3) \quad \forall a_1, a_2, a_3 \in A.$$

For $a_1, a_2 \in A$, it easily follows from (4) that the group-theoretic commutator $[\bar{a}_1, \bar{a}_2] \in F^*$. The following identities thus follow from the basic properties of commutators (see, for example, [Ro, Section 5.1.5]):

$$(5) \quad [\bar{a}_1 \bar{a}_2, \bar{a}_3] = [\bar{a}_1, \bar{a}_3] [\bar{a}_2, \bar{a}_3],$$

$$(6) \quad [\bar{a}_1, \bar{a}_2 \bar{a}_3] = [\bar{a}_1, \bar{a}_2] [\bar{a}_1, \bar{a}_3],$$

$$(7) \quad [\bar{a}_1, \bar{a}_2^{-1}] = [\bar{a}_1, \bar{a}_2]^{-1},$$

$$(8) \quad [\bar{a}_1^{-1}, \bar{a}_2] = [\bar{a}_1, \bar{a}_2]^{-1}$$

For a subset X of A , we define $\overline{X} = \{\bar{x} : x \in X\}$. If $X, Y \subset A$, we set

$$[\overline{X}, \overline{Y}] = \langle [\bar{x}, \bar{y}] : x \in X, y \in Y \rangle.$$

Thus $[\overline{X}, \overline{Y}]$ is a subgroup of F^* . If $\alpha \in F * A$, we may express $\alpha = \sum_{a \in A} \lambda_a \bar{a}$, where $\lambda_a \in F$ and $\lambda_a = 0$ for “almost all” $a \in A$. We define the support of α (in A) as

$$\text{Supp}(\alpha) = \{a \in A : \lambda_a \neq 0\}.$$

Note that for a subgroup B of A , the subalgebra generated by $\overline{B} \subset F * A$ is a twisted group algebra $F * B$. It was shown in [MP, Proposition 1.3] that an algebra $F * A$ is simple if and only if it has center F .

Proposition 2.1. *An algebra $F * A$ has center exactly F if and only if for each subgroup $A_1 < A$ with finite index $F * A_1$ has center F .*

Proof. Suppose that $F * A$ has center F . Let $A_1 \leq A$ be a subgroup such that $l := [A : A_1] < \infty$. We claim that $F * A_1$ also has center F . Using [MP, Proposition 1.3], we may assume that \bar{a}_1 is central in $F * A_1$ for $1 \neq a_1 \in A_1$. For any $a \in A$, (5) and (6) yield:

$$[\bar{a}_1^l, \bar{a}] = [\bar{a}_1, \bar{a}]^l = [\bar{a}_1, \bar{a}^l] = 1,$$

where the last equality holds since $a^l \in A_1$. Since A is torsion-free by definition, $1 \neq a_1^l$. Thus \bar{a}_1^l is a nonscalar central element of $F * A$. The converse is clear. \square

It is well known (see, for example, [Pa2, Lemma 37.8]) that for each subgroup $B \leq A$,

$$\mathcal{S}_B := F * B \setminus \{0\}$$

is an Ore subset in $F * A$. Thus $F * A$ has a localization $(F * A)\mathcal{S}_B^{-1}$ at \mathcal{S}_B which has the structure of a *crossed product* $D_B * A/B$ of A/B over the quotient division ring D_B of $F * B$.

A crossed product $D * C$ of an abelian group C over a division ring D is an associative ring which has a copy $\overline{C} := \{\bar{c} \mid c \in C\}$ of C as D -basis with the multiplication satisfying

$$\begin{aligned} \bar{c}_1 \bar{c}_2 &= \tau(c_1, c_2) \overline{c_1 c_2} & \forall c_1, c_2 \in C, \\ \bar{c} d &= \sigma_c(d) \bar{c}, & \forall c \in C, \forall d \in D, \end{aligned}$$

for a suitable function $\tau : C \times C \rightarrow D^*$ and automorphisms $\sigma_c \in \text{Aut}(D)$. We note that twisted group algebras are special types of crossed products arising when the division ring D lies in the center. We refer to [Pa2] for further details on crossed products.

3. GK DIMENSION AND CRITICAL MODULES

All modules that we shall consider shall be right modules. In our investigations we shall have to consider modules over suitable localizations of the algebras $F * A$ as just discussed. Thus we shall now briefly discuss the basic module theory of crossed products $D * A$, where D is a division ring.

Let M be a finitely generated $D * A$ -module and $B \leq A$ be a subgroup. It is known that $\mathcal{S}_B := D * B \setminus \{0\}$ is an Ore subset (e.g., [Pa2, Lemmma 37.8]) and it easily follows from this that

$$T_B(M) := \{m \in M \mid m\beta = 0 \text{ for some } \beta \in \mathcal{S}_B\},$$

is an $F * A$ -submodule of M known as the $F * B$ -torsion submodule of M . We shall say that M is $F * B$ -torsion-free if $T_B(M) = 0$ and $F * B$ -torsion if $T_B(M) = M$.

In [BG2], a dimension for $D * A$ -modules was introduced and was shown to coincide with the standard Gelfand-Kirillov dimension (GK dimension) measured over D . The next proposition characterizes the GK dimension for finitely generated $D * A$ -modules.

Proposition 3.1 (Brookes and Groves). *Let M be a finitely generated $D * A$ -module. Then $\mathcal{GK}(M)$ equals the supremum of the ranks of subgroups B of A such that M is not torsion as $D * B$ -module. Furthermore, let a_1, \dots, a_n freely generate A and define*

$$\mathcal{F} = \{\langle X \rangle : X \subset \{a_1, \dots, a_n\}\}$$

*with the convention that $\langle \emptyset \rangle = \langle 1 \rangle$. Then $\mathcal{GK}(M)$ is simply the supremum of the ranks of subgroups B in \mathcal{F} such that M is not $D * B$ -torsion.*

Remark 3.2. *Let us note some consequences. Let $\{x_1, \dots, x_n\}$ be a basis of A . Let M be a finitely generated $F * A$ -module with $\mathcal{GK}(M) = r$, where $0 < r < \text{rk}(A)$. By the preceding proposition, there must be a nonempty subset $\{i_1, i_2, \dots, i_r\} \subset \{1, 2, \dots, n\}$ so that M is not $F * \langle x_{i_1}, x_{i_2}, \dots, x_{i_r} \rangle$ -torsion. Let $\{j_1, \dots, j_{n-r}\}$ be the (set) complement to $\{i_1, i_2, \dots, i_r\}$ in $\{1, \dots, n\}$. and set $S := F * \langle x_{i_1}, x_{i_2}, \dots, x_{i_r} \rangle \setminus \{0\}$. As already noted, S is an Ore subset in $F * A$ and the localization $(F * A)S^{-1}$ is a crossed product $D * \langle x_{j_1}, \dots, x_{j_{n-r}} \rangle$, where D is the quotient division ring of $F * \langle x_{i_1}, x_{i_2}, \dots, x_{i_r} \rangle$. For convenience, we can assume that $\{i_1, i_2, \dots, i_r\} = \{1, 2, \dots, r\}$*

Following the notation in [MP], we shall write $F(i_1, i_2, \dots, i_r)$ for D and denote $D * \langle x_{j_1}, \dots, x_{j_{n-r}} \rangle$ as

$$F(x_{i_1}, x_{i_2}, \dots, x_{i_r})[x_{j_1}, \dots, x_{j_{n-r}}].$$

Lemma 3.3. *With the above notation, MS^{-1} is a nonzero $D * \langle x_{j_1}, \dots, x_{j_{n-r}} \rangle$ -module that is finite dimensional as D -vector space.*

Proof. By [BG2, Lemma 2.2(3)], if the GK dimension of MS^{-1} (over D) is 0, then MS^{-1} is finite dimensional as D -space. In view of Proposition 3.1 it thus suffices to show that MS^{-1} is $D * \langle x_{j_l} \rangle$ -torsion for all $l \in \{1, \dots, n-r\}$. We set $A_l = \langle x_{i_1}, x_{i_2}, \dots, x_{i_r}, x_{j_l} \rangle$. In view of Proposition 3.1, M is $F * A_l$ -torsion. Thus if $m \in M$ then $m\alpha_l = 0$ for some nonzero $\alpha_l \in F * A_l$. If $\phi : M \rightarrow MS^{-1}$ is the canonical homomorphism then each element of MS^{-1} may be expressed as $\phi(m)s^{-1}$ for some $m \in M$ and $s \in S$. But then

$$\begin{aligned} \phi(m)s^{-1}(s\alpha_l) &= \phi(m)(s^{-1}s\alpha_l) \\ &= \phi(m)\alpha_l \\ &= \phi(m\alpha_l) \\ &= \phi(0) \\ &= 0. \end{aligned}$$

Now $s\alpha_l \in F * A_l \subset D * \langle x_{j_l} \rangle$. Hence MS^{-1} is $D * \langle x_{j_l} \rangle$ -torsion. The lemma now follows. \square

We next note a useful property of the GK dimension for $D * A$ -modules was shown in [BG2, Lemma 2.2].

Proposition 3.4 (Brookes and Groves). *Let M be a finitely generated $D * A$ -module. If $B \leq A$ is a subgroup and N is a finitely generated $D * B$ -submodule of M , then $\mathcal{GK}(N) \leq \mathcal{GK}(M)$. Furthermore, if $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence of $D * A$ -modules, then*

$$\mathcal{GK}(M) = \sup\{\mathcal{GK}(M'), \mathcal{GK}(M'')\}.$$

The GK dimension of a $D * A$ -module does not change in passing to a subgroup of finite index.

Proposition 3.5. *Let M be a finitely generated $D * A$ -module with $\mathcal{GK}(M) = r$, where $0 \leq r \leq \text{rk}(A)$. If $A' < A$ is a subgroup of finite index in A and M' is the $D * A'$ -module structure on M , then $\mathcal{GK}(M') = r$.*

Proof. By Proposition 3.1, A has a subgroup B with rank r so that M is not $D * B$ -torsion. Set $B' = A' \cap B$. By the hypothesis in the proposition, $[A : A'] < \infty$ and hence $\text{rk}(B') = r$. Clearly M' is not torsion as $D * B'$ -module. Hence $\mathcal{GK}(M') \geq r$ by Proposition 3.1. Moreover $\mathcal{GK}(M') \leq r$ by Proposition 3.4. \square

In considering $D * A$ -modules, it is often useful to pass to a *critical* submodule. Critical $D * A$ modules were introduced in [BG2] and certain facts concerning them were also established (see [BG2, Section 2]) that account for their usefulness.

Definition 3.6. *A nonzero $D * A$ -module M is critical if for any nonzero proper submodule N , $\mathcal{GK}(M) > \mathcal{GK}(M/N)$.*

Proposition 3.7 (Proposition 2.5 of [BG2]). *Every nonzero $D * A$ -module contains a critical submodule.*

Proposition 3.8. *A critical $D * A$ -module of minimum possible GK dimension is simple*

Proof. Let N be such a module and let L be a nonzero proper submodule of N . Then $\mathcal{GK}(N/L) < \mathcal{GK}(N)$. However, this contradicts the definition of a critical module. \square

Lemma 3.9. *Let N be a simple $F * A$ -module which is not torsion as $\mathcal{C} := F * \langle x_1, \dots, x_r \rangle$ -module, where $\{x_1, \dots, x_n\}$ denotes a basis of A and $0 < r < n$. Let*

$$F(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$$

*denote the right Ore localization of $F * A$ at $\mathcal{C} \setminus \{0\}$ and let N_r stand for the corresponding localization of N . Then*

$$\mathcal{GK}(N_r) = \mathcal{GK}(N) - r,$$

where the GK dimension of N_r is measured relative to the division ring $F(x_1, \dots, x_r)$.

Proof. We note that N must be a torsion-free \mathcal{C} -module. Indeed since N is simple by the hypothesis the \mathcal{C} -torsion submodule of N is either N or 0. But the former possibility is ruled out since N is not a torsion \mathcal{C} -module by the hypothesis.

Now N satisfies the hypothesis of [BG3, Lemma 4.5(ii)] which asserts that N_r is critical. However, in the proof it is actually shown that N_r is critical with $\mathcal{GK}(N_r) = \mathcal{GK}(N) - r$. \square

4. THE BROOKES–GROVES GEOMETRIC INVARIANT

We now continue our discussion of the module theory of crossed products $D * A$ by describing the geometric invariant of Brookes and Groves that can be associated with each finitely generated $D * A$ -module. This invariant was modelled on the original Bieri–Strebel invariant which was used to give a geometric criterion for a meta-abelian group to be finitely presented. Using this invariant we shall establish

certain facts concerning $D * A$ -modules which will be used in our investigations that follow.

Let M be a finitely generated $D * A$ -module. The dual $A^* := \text{Hom}_{\mathbb{Z}}(A, \mathbb{R})$ of A is easily seen to be a real vector space of dimension equal to the rank of A . Hence we may identify A^* with \mathbb{R}^n where $n = \text{rk}(A)$ and speak of group characters $\phi \in A^*$ as points. A point $\phi \in A^*$ determines a submonoid $A(0, \phi)$ and a subsemigroup $A(+, \phi)$ of A as follows:

$$\begin{aligned} A(0, \phi) &:= \{a \in A \mid \phi(a) \geq 0\} \\ A(+, \phi) &:= \{a \in A \mid \phi(a) > 0\}. \end{aligned}$$

For a subset X of A , we shall denote the subset of $F * A$ of all elements α with $\text{Supp}(\alpha) \in X$ by $F * X$. With each point $\phi \in A^*$ a module for $F * \ker \phi$ may be associated as follows.

Definition 4.1. *Let M be a finitely generated $D * A$ -module and let \mathcal{X} be a (finite) generating set for M . For a point $\phi \in A^*$, the trailing coefficient module $TC_{\phi}(M)$ of M at ϕ is defined as*

$$TC_{\phi}(M) = \mathcal{X}(D * A(0, \phi)) / \mathcal{X}(D * A(+, \phi)).$$

It is easily seen that $TC_{\phi}(M)$ is a finitely generated $D * \ker \phi$ -module. In general, $TC_{\phi}(M)$ may depend on the choice of a generating set \mathcal{X} for M . However, the next definition turns out to be independent of such a choice.

Definition 4.2. *Let M be finitely generated $D * A$ -module. Then $\Delta(M)$ is defined to be the subset of all $\phi \in A^*$ so that $TC_{\phi}(M) = 0$.*

Remark 4.3. *Since $A(0, 0) = A$ and $A(+, 0)$ is empty, hence $TC_0(M) \neq 0$ if M is nonzero. It follows that $0 \in \Delta(M)$ for nonzero M .*

As already noted, $A^* \cong \mathbb{R}^n$ and so $\Delta(M)$ can be identified with a subset of \mathbb{R}^n . A *convex polyhedron* in \mathbb{R}^n is an intersection of a finite number of closed linear half spaces in \mathbb{R}^n . A *polyhedron* is a union of finitely many convex polyhedra. Suppose that a basis is fixed in A . A subspace of A^* is *rationally defined* if it has a set of generators each of which is a rational linear combination of the elements of the dual basis in A^* . A convex polyhedron is rational if it can be defined using half spaces with a rational subspace as boundary. A polyhedron is rational if it is a finite union of rational convex polyhedra. The dimension of a convex polyhedron is the dimension of the subspace spanned by it. The dimension of a polyhedron is the maximum of the dimensions of its constituent convex polyhedra.

It was shown in [BG1] that for a special class of $D * A$ -modules $\Delta(M)$ is a closed rational polyhedron and a weak polyhedrality result was obtained in [BG2] for arbitrary finitely generated $D * A$ -modules. Later it was shown in [Wa] that $\Delta(M)$ is a closed rational polyhedral cone for all finitely generated $D * A$ -modules M .

Given a subgroup $B \leq A$, the inclusion mapping $\iota : B \hookrightarrow A$ induces the restriction mapping $\text{res}_B : A^* \rightarrow B^*$. It is easily seen that the rational subspaces of A^* are precisely the kernels of the maps res_B where B ranges over all subgroups of A .

For a subset S of \mathbb{R}^n and a point $x \in S$, a *neighborhood* of x in S is the intersection with S of a ball in \mathbb{R}^n centered on x . The intersection of a ball in \mathbb{R}^n with an m -dimensional subspace will be called an m -ball. A point x of S will be called *regular* if some neighborhood of x in S is an m -ball and S has no points with this property for a larger choice of m . For any subset S of \mathbb{R}^n , the *essential part*

of S^* is the (Euclidean) closure of the set of the regular points of S . With this notation, Theorem 4.4 of [BG2] asserts that for a finitely generated $D * A$ -module M , $\Delta^*(M)$ is a closed rational polyhedron of dimension equal to the GK dimension of M . The vector spans of neighborhoods of the regular points in $\Delta^*(M)$ are the *carrier spaces* of $\Delta^*(M)$. Note that each carrier space has dimension equal to m , where $m = \mathcal{GK}(M)$. A carrier space V of $\Delta^*(M)$ is rational (see [G]) and $V = \ker \text{res}_B$ for some isolated subgroup $B \leq A$, that is a subgroup $B \leq A$ such that A/B is torsion-free. We also note that in this case

$$\dim(V) + \text{rk}(B) = \text{rk}(A),$$

whence

$$m + \text{rk}(B) = \text{rk}(A).$$

Now we shall employ the geometric invariant to obtain some results concerning $D * A$ -modules that we shall need later.

The following theorem was established in a special case in [GU]. The reasoning given here is due to [BG4].

Theorem 4.4. *Let M be a finitely generated $F * A$ -module with $\mathcal{GK}(M) = r$, where $0 < r \leq \text{rk}(A) - 1$. Let $\{x_1, \dots, x_n\}$ be a basis of A and suppose that M is not torsion as $F * \langle x_1, \dots, x_r \rangle$ -module. Then there exist $l_{r+1}, \dots, l_n \in \langle x_1, \dots, x_r \rangle$ and nonzero integers s_{r+1}, \dots, s_n so that*

$$[\bar{l}_j \bar{x}_j^{s_j}, \bar{l}_k \bar{x}_k^{s_k}] = 1 \quad \forall j, k \in \{r+1, \dots, n\}.$$

Remark 4.5. *The hypothesis in this theorem is equivalent to the localization*

$$F(x_1, \dots, x_r)[x_{r+1}, \dots, x_n]$$

having a nonzero finite dimensional module over the division ring $F(x_1, \dots, x_r)$ (See [B, Section 2]).

Proof. Set $B = \langle x_1, \dots, x_r \rangle$. It suffices to show that there exists a subgroup $E < A$ such that $B \cap E = \langle 1 \rangle$ and $F * E$ is commutative. By the hypothesis the $F * B$ -torsion submodule $T_B(M)$ is a proper submodule of M . Moreover $N := M/T_B(M)$ is $F * B$ -torsion-free. Any nonzero submodule of N will also be torsion-free over $F * B$ and so we may assume noting Proposition 3.7 that N is critical.

By [BG2, Theorem 5.5] and $\pi_B \Delta(N) = \Delta(L)$ for some cyclic critical $F * B$ -submodule L of minimum dimension. Since L is $F * B$ -torsion-free $\pi_B \Delta(N) = B^*$. Thus $\pi_B(\Delta^*(N)) = B^*$. Since the real vector space B^* cannot be a union of a finite many proper subspaces, hence at least one of the r -dimensional spaces spanned by the convex polyhedra in $\Delta^*(N)$ is mapped surjectively onto B^* . Let V be such a subspace. Since C^* is also r -dimensional, hence

$$(9) \quad \ker \pi_B + V = \ker \pi_B \oplus V = A^*.$$

Since V is a rational subspace, $V = \ker \pi_C$ for some $C \leq A$ such that C . Moreover C has a subgroup E of finite index such that $F * E$ is commutative. It easily follows from (9) that $B \cap E = \langle 1 \rangle$. \square

Lemma 4.6. *Let M be a nonzero finitely generated $D * A$ -module with $\mathcal{GK}(M) \geq 1$. Let $C < A$ be an infinite cyclic subgroup of A such that M is $D * C$ -torsion. Let V be a carrier space of $\Delta^*(M)$. Then $V^\circ \cap C \neq \langle 1 \rangle$.*

Proof. Set $B_V = V^\circ$. Suppose that $B_V \cap C = \langle 1 \rangle$. It is known (e.g., [G, Section 1]) that the convex polyhedral cone in $\Delta^*(M)$ which spans V contains a point ψ so that $\ker \psi = B_V$. Since ψ is a point of $\Delta(N)$, $TC_\psi(N) \neq 0$. Let \mathcal{X} be a finite generating set for N . It is easily seen from the definition of $TC_\psi(N)$ that there exists $x \in \mathcal{X}$ so that

$$(10) \quad x \notin \mathcal{X}(D * A(+)).$$

By the hypothesis in the lemma M is $D * C$ -torsion and so there exists $\gamma \in F * C \setminus \{0\}$ so that $x\gamma = 0$. Moreover if $c_i, c_j \in \text{Supp}(\gamma)$ are distinct elements then $\psi(c_i) \neq \psi(c_j)$. For otherwise,

$$c_i c_j^{-1} \in C \cap \ker \psi = C \cap B_V$$

contrary to our assumption that $B_V \cap C = \langle 1 \rangle$. Hence multiplying by a suitable unit in $F * C$ we may assume that $\gamma = 1 - \gamma_\psi$, where $\gamma_\psi \in D * C \cap D * A(+)$. Then we have $x = x\gamma_\psi \in \mathcal{X}(D * A(+))$ contrary to (10).

Hence $B_V \cap C \neq \langle 1 \rangle$. □

5. THE DIMENSION OF A TENSOR PRODUCT

The tensor product $F * A_1 \otimes_F F * A_2$ of twisted group algebras is a twisted group algebra of $A_1 \times A_2$ over F . We shall write \otimes for \otimes_F when there is no danger of confusion. We begin with the following characterization of $\dim(F * A)$ which was conjectured in [MP, Section 3.3] and was shown in [B, Theorem A].

Theorem 5.1 (C.J.B. Brookes). *The dimension of an algebra $F * A$ equals the supremum of the ranks of subgroups $B \leq A$ so that $F * B$ is commutative.*

We note a few consequences of the above theorem.

Corollary 5.2. *Given an algebra $F * A$, let $A' < A$ be a subgroup of finite index in A . Then*

$$\dim(F * A') = \dim(F * A).$$

Corollary 5.3. *Let $F * A_1$ and $F * A_2$ be arbitrary twisted group algebras. Then*

$$(11) \quad \dim(F * A_1 \otimes_F F * A_2) \geq \dim(F * A_1) + \dim(F * A_2).$$

In Theorem 5.7 we shall show an upper bound for $\dim(F * A_1 \otimes_F F * A_2)$. We shall need the next few facts in the proof of this theorem. The following fundamental result was shown in [B, Theorem 3].

Theorem 5.4 (Brookes). *If an algebra $F * A$ has a finitely generated nonzero module N with $\mathcal{GK}(N) = m$, then A contains a subgroup B with rank $\text{rk}(A) - m$ such that $F * B$ is commutative.*

Corollary 5.5. *Let M be a nonzero finitely generated module over an algebra $F * A$. Then $\mathcal{GK}(M) \geq \text{rk}(A) - \dim(F * A)$.*

Proof. Indeed if this were not true then by the last theorem, A would contain a subgroup B with $\text{rk}(B) > m$ so that $F * B$ is commutative. But this contradicts Theorem 5.1. □

We shall use the last result in the following form.

Lemma 5.6. *If an algebra $F * A$ has dimension m then it has a simple module N with $\mathcal{GK}(N) = \text{rk}(A) - m$.*

Proof. It was shown in [B, Section 2] that there exists a nonzero finitely generated $F * A$ -module M with $\mathcal{GK}(M) = \text{rk}(A) - m$. Here we merely argue that such a module M has a simple submodule whose GK dimension must be $\text{rk}(A) - m$ in view of Corollary 5.5. But in view of [MP, Lemma 5.6] and Corollary 5.5, M has finite length. \square

We now prove our first main result.

Theorem 5.7. *Given algebras $F * A_1$ and $F * A_2$ suppose that $\dim(F * A_i) < \text{rk}(A_i)$ for $i \in \{1, 2\}$. Let $d := \dim(F * A_1 \otimes_F F * A_2)$*

$$(12) \quad d \leq \min\{\dim(F * A_1) + \text{rk}(A_2), \dim(F * A_2) + \text{rk}(A_1)\} - 1.$$

Proof. Without loss of generality, we may assume that

$$\dim(F * A_1) + \text{rk}(A_2) \leq \dim(F * A_2) + \text{rk}(A_1).$$

We will show that

$$\dim(F * A_1 \otimes_F F * A_2) \leq \dim(F * A_1) + \text{rk}(A_2) - 1.$$

Set $r_i = \text{rk}(A_i)$, $d_i = \dim(F * A_i)$ for $i \in \{1, 2\}$, $d = \dim(F * A_1 \otimes_F F * A_2)$ and $\mathcal{A} = F * A_1 \otimes_F F * A_2$.

If $d > d_1 + r_2$, then by Lemma 5.6, \mathcal{A} has a nonzero finitely generated module M with $\mathcal{GK}(M) < r_1 - d_1$. Let M_1 be a nonzero $F * A_1$ -submodule of M . Then $\mathcal{GK}(M_1) \leq \mathcal{GK}(M)$ by Proposition 3.4. But this is impossible in view of Corollary 5.5.

We now suppose that $d = d_1 + r_2$. In this case, by Lemma 5.6, \mathcal{A} has a simple module M' with $\mathcal{GK}(M') = r_1 - d_1$. Let M'_1 be a finitely generated $F * A_1$ -submodule of M . In view of Corollary 5.5 and Proposition 3.4 we have

$$r_1 - d_1 \leq \mathcal{GK}(M'_1) \leq \mathcal{GK}(M') = r_1 - d_1,$$

whence $\mathcal{GK}(M'_1) = r_1 - d_1$. Let $\{x_1, \dots, x_{r_1}\}$ be a basis in A_1 . By Proposition 3.1, there must be a subset $\{j_1, \dots, j_{r_1-d_1}\} \subset \{1, \dots, r_1\}$ so that M'_1 and hence M' is not $F * \langle x_{j_1}, \dots, x_{j_{r_1-d_1}} \rangle$ -torsion. There is no harm in assuming that $(j_1, \dots, j_{r_1-d_1}) = (1, \dots, r_1 - d_1)$. By Lemma 3.3, the crossed product

$$F(x_1, \dots, x_{r_1-d_1})[x_{r_1-d_1+1}, \dots, x_{r_1}, y_1, \dots, y_{r_2}]$$

has a nonzero module finite dimensional as $F(x_1, \dots, x_{r_1-d_1})$ -space. Moreover by Theorem 4.4, there exist

$$l_i, l'_j \in \langle x_1, \dots, x_{r_1-d_1} \rangle \text{ and } s_i, s'_j \in \mathbb{Z} - \{0\} \quad \forall i \in \{r_1-d_1+1, \dots, r_1\}, j \in \{1, \dots, r_2\}$$

such that

$$(13) \quad [\bar{l}_i \bar{x}_i^{s_i}, \bar{l}_k \bar{x}_k^{s_k}] = 1 \quad \forall i, k \in \{r_1 - d_1 + 1, \dots, r_1\},$$

$$(14) \quad [\bar{l}_i \bar{x}_i^{s_i}, \bar{l}'_j \bar{y}_j^{s'_j}] = 1 \quad \forall i \in \{r_1 - d_1 + 1, \dots, r_1\}, j \in \{1, \dots, r_2\}, \text{ and}$$

$$(15) \quad [\bar{l}'_j \bar{y}_j^{s'_j}, \bar{l}'_p \bar{y}_p^{s'_p}] = 1 \quad \forall j, p \in \{1, \dots, r_2\}.$$

We note that (13) means that for

$$C_1 := \langle l_{r_1-d_1+1} x_{r_1-d_1+1}^{s_{r_1-d_1+1}}, \dots, l_{r_1} x_{r_1}^{s_{r_1}} \rangle$$

the subalgebra $F * C_1$ is commutative. Since \mathcal{A} is a tensor product of $F * A_1$ and $F * A_2$, we have

$$(16) \quad [\bar{l}_i \bar{x}_i^{s_i}, \bar{y}_j^{s'_j}] = 1 \quad \forall i \in \{r_1 - d_1 + 1, \dots, r_1\}, j \in \{1, \dots, r_2\}.$$

Applying (16) and (6) to (14) we get

$$(17) \quad 1 = [\bar{l}_i \bar{x}_i^{s_i}, \bar{l}_j' \bar{y}_j^{s_j'}]$$

$$(18) \quad = [\bar{l}_i \bar{x}_i^{s_i}, \bar{l}_j'] [\bar{l}_i \bar{x}_i^{s_i}, \bar{y}_j^{s_j'}]$$

$$(19) \quad = [\bar{l}_i \bar{x}_i^{s_i}, \bar{l}_j'] \quad \forall i \in \{r_1 - d_1 + 1, \dots, r_1\}, j \in \{1, \dots, r_2\}.$$

By Theorem 5.1, the subgroup C_1 of A_1 has the maximal possible rank with respect to $F * C_1$ being commutative. But then (17) – (19) imply that

$$l_j'^{t_j} \in C_1 \quad \forall j \in \{1, \dots, r_2\},$$

where t_j is a positive integer for each j . We now use the fact that \mathcal{A} is a tensor product and apply (5)–(6) to (15). We get

$$\begin{aligned} 1 &= [\bar{l}_j' \bar{y}_j^{s_j'}, \bar{l}_p' \bar{y}_p^{s_p'}] \\ &= [\bar{l}_j', \bar{l}_p'] [\bar{y}_j^{s_j'}, \bar{y}_p^{s_p'}] \\ &= ([\bar{l}_j', \bar{l}_p'] [\bar{y}_j^{s_j'}, \bar{y}_p^{s_p'}])^{t_j t_p} \\ &= [\bar{l}_j', \bar{l}_p']^{t_j t_p} [\bar{y}_j^{s_j'}, \bar{y}_p^{s_p'}]^{t_j t_p} \\ &= [\bar{l}_j'^{t_j}, \bar{l}_p'^{t_p}] [\bar{y}_j^{s_j' t_j}, \bar{y}_p^{s_p' t_p}] \\ &= [\bar{y}_j^{s_j' t_j}, \bar{y}_p^{s_p' t_p}] \quad \forall j, p \in \{1, \dots, r_2\}, \end{aligned}$$

where in the last step we have made use of the commutativity of $F * C$. We have just shown that

$$[\bar{y}_j^{s_j' t_j}, \bar{y}_p^{s_p' t_p}] = 1 \quad \forall j, p \in \{1, \dots, r_2\},$$

where s_j' and t_j are nonzero integers for all j . It follows that A_2 has a subgroup A_2' of finite index so that $F * A_2'$ is commutative. By Theorem 5.1, $d_2 = r_2$ contrary to the hypothesis. \square

Remark 5.8. *If we drop the assumption $\dim(F * A_i) < \text{rk}(A_i)$ from the preceding theorem, the theorem may no longer hold true. For example, suppose that $\text{rk}(A_i) = 2$ for $i \in \{1, 2\}$, $\dim(F * A_1) = 2$ and $\dim(F * A_2) = 1$. By Corollary 5.3, $\dim(F * A_1 \otimes_F F * A_2) \geq 3$. But*

$$\min\{\dim(F * A_1) + \text{rk}(A_2) - 1, \dim(F * A_2) + \text{rk}(A_1) - 1\} = \min\{3, 2\} = 2.$$

Remark 5.9. *The hypotheses $\dim(F * A_i) < \text{rk}(A_i)$ is used only in the second part of the proof to show that $d \neq d_1 + r_2$. The fact*

$$\dim(F * A_1 \otimes F * A_2) \leq \min\{\dim(F * A_1) + \text{rk}(A_2), \dim(F * A_2) + \text{rk}(A_1)\}$$

*remains valid in the case $\dim(F * A_i) = \text{rk}(A_i)$ for some $i \in \{1, 2\}$.*

The next corollary shows that if atleast one of the algebras $F * A_i$ is “virtually commutative” then the dimension is additive.

Corollary 5.10. *Given twisted group algebras $F * A_1$ and $F * A_2$, if $\dim(F * A_i) = \text{rk}(A_i)$ for some $i \in \{1, 2\}$ then equality holds in (5.3).*

Proof. Without loss of generality, we may assume that $\dim(F * A_i) = \text{rk}(A_i)$. As in Remark 5.9,

$$\begin{aligned} \dim(F * A_1 \otimes F * A_2) &\leq \min\{\dim(F * A_1) + \text{rk}(A_2), \dim(F * A_2) + \text{rk}(A_1)\} \\ &= \min\{\text{rk}(A_1) + \text{rk}(A_2), \dim(F * A_2) + \dim(F * A_1)\} \\ &= \dim(F * A_1) + \dim(F * A_2). \end{aligned}$$

Hence equality holds in (11). \square

Equality holds in (11) in the following situation also.

Corollary 5.11. *Let $F * A_1$ and $F * A_2$ be twisted group algebras such that $\dim(F * A_i) < \text{rk}(A_i)$. If $\dim(F * A_i) = \text{rk}(A_i) - 1$ for some $i \in \{1, 2\}$, then*

$$\dim(F * A_1 \otimes_F F * A_2) = \dim(F * A_1) + \dim(F * A_2).$$

Proof. Without loss of generality we assume that $\dim(F * A_1) = \text{rk}(A_1) - 1$. We set $r_i = \text{rk}(A_i)$, $d_i = \dim(F * A_i)$ for $i \in \{1, 2\}$ and $d = \dim(F * A_1 \otimes_F F * A_2)$. By Theorem 5.7,

$$\begin{aligned} (20) \quad d &\leq \min\{d_1 + r_2, d_2 + r_1\} - 1 \\ (21) \quad &= \min\{r_1 + r_2 - 1, d_2 + r_1\} - 1 \\ (22) \quad &= d_2 + r_1 - 1 \\ (23) \quad &= d_2 + d_1. \end{aligned}$$

On the other hand $d \geq d_1 + d_2$ by Corollary 5.3. Hence $d = d_1 + d_2$.

We recall our definition of $\text{co-dim}(F * A)$ given in Section 1 as

$$\text{co-dim}(F * A) = \text{rk}(A) - \dim(F * A).$$

With this the last two corollaries may be combined in the following form.

Corollary 5.12. *Given algebras $F * A_1$ and $F * A_2$ suppose that $\text{co-dim}(F * A_i) \leq 1$ holds for some $i \in \{1, 2\}$. Then*

$$\dim(F * A_1 \otimes_F F * A_2) = \dim(F * A_1) + \dim(F * A_2).$$

We recall that an algebra $\otimes_{i=1}^k F * B_i$, where $B_i \cong \mathbb{Z}^2$ is known as the multiplicative analogue of the Weyl algebra [J]. As $1 \leq \dim(F * B_i) \leq 2$ (Section 1) we obtain the following.

Corollary 5.13. *For any algebras $F * B_j$, where $B_j \cong \mathbb{Z}^2$ and $j \in \{1, \dots, s\}$,*

$$\dim(F * B_1 \otimes F * B_2 \otimes \dots \otimes F * B_s) = \sum_{j=1}^s \dim(F * B_j).$$

5.1. The inequality becomes strict. As already noted in the example in Section 1, the upper bound in Theorem 5.7 is attained for the tensor products in which each tensor factor has dimension one. It easily follows from Corollary 5.12 that this upper bound is also attained in the case when both the (tensor) factors have codimension one.

We shall now prove (see Theorem 5.14) that in the case both the dimension and the codimension of each (tensor) factor exceeds one the inequality in Theorem 5.7 is strict.

Theorem 5.14. *Let $F * A_1$ and $F * A_2$ be twisted group algebras such that $\text{co-dim}(F * A_i) \geq 2$, $\dim(F * A_i) \geq 2$ and $F * A_i$ has center F . Let $d := \dim(F * A_1 \otimes F * A_2)$. Then*

$$(24) \quad d < \min\{\dim(F * A_1) + \text{rk}(A_2), \dim(F * A_2) + \text{rk}(A_1)\} - 1.$$

Proof. Set $\mathcal{A} = F * A_1 \otimes F * A_2$, $n_i = \text{rk}(A_i)$ and $d_i = \dim(F * A_i)$, where $i \in \{1, 2\}$. Let $\{x_1, \dots, x_{n_1}\}$ and $\{y_1, \dots, y_{n_2}\}$ be bases in A_1 and A_2 respectively.

Without loss of generality, we may assume that

$$\text{co-dim}(F * A_1) \geq \text{co-dim}(F * A_2).$$

In other words, $n_1 - d_1 \geq n_2 - d_2$ and so $n_1 + d_2 \geq n_2 + d_1$. Consequently the right side of (24) is $n_2 + d_1 - 1$. We must thus show that

$$(25) \quad \dim(\mathcal{A}) < n_2 + d_1 - 1.$$

But in view of Theorem 5.7, it suffices to show that

$$\dim(\mathcal{A}) \neq n_2 + d_1 - 1.$$

To this end we suppose that

$$\dim(\mathcal{A}) = n_2 + d_1 - 1.$$

Applying Lemma 5.6, \mathcal{A} has a simple module N with

$$\mathcal{GK}(N) = n_1 - d_1 + 1.$$

Using Proposition 3.7, let N_1 be a finitely generated critical $F * A_1$ -submodule of N of the maximum possible GK dimension. In view of Corollary 5.5 and Proposition 3.4,

$$(26) \quad n_1 - d_1 \leq \mathcal{GK}(N_1) \leq n_1 - d_1 + 1.$$

We first assume that

$$(27) \quad \mathcal{GK}(N_1) = n_1 - d_1.$$

By Proposition 3.1 and Remark 3.2, N_1 and hence N is not $F * \langle x_1, x_2, \dots, x_{n_1-d_1} \rangle$ -torsion. This means that we may localize $n_1 - d_1$ generators say $x_1, \dots, x_{n_1-d_1}$ of A_1 . We then obtain the crossed product

$$F(x_1, \dots, x_{n_1-d_1})[x_{n_1-d_1+1}, \dots, x_{n_1}, y_1, \dots, y_{n_2}].$$

Set $\hat{A}_2 = \langle x_{n_1-d_1+1}, \dots, x_{n_1}, y_1, \dots, y_{n_2} \rangle$ and note that $\text{rk}(\hat{A}_2) > \text{rk}(A_2)$. By Lemma 3.9, the corresponding localization $N_{n_1-d_1}$ of N has GK dimension 1 (over $F(x_1, \dots, x_{n_1-d_1})$) since

$$\mathcal{GK}(N) = n_1 - d_1 + 1.$$

Let V be a carrier space of $\Delta^*(N_{n_1-d_1})$. Then

$$\dim(V) = \mathcal{GK}(N_{n_1-d_1}) = 1.$$

Let B_V be the isolated subgroup of \hat{A}_2 so that $V = B_V^\circ$. Then $\text{rk}(B_V) = \text{rk}(\hat{A}_2) - 1$.

We note that $N_{n_1-d_1}$ must be $F(x_1, \dots, x_{n_1-d_1})[x_j]$ -torsion for each $j \in \{n_1 - d_1 + 1, \dots, n_1\}$ otherwise N would not be $F * \langle x_1, \dots, x_{n_1-d_1}, x_j \rangle$ -torsion. But this is a contradiction to (27) in view of Proposition 3.1.

Thus by Lemma 4.6, there exist nonzero integers s_j so that $x_j^{s_j} \in B_V$ for all $j \in \{n_1 - d_1 + 1, \dots, n_1\}$. Since $\text{rk}(B_V) = \text{rk}(\hat{A}_2) - 1$, hence

$$\begin{aligned} \text{rk}(B_V \cap A_2) &\geq \text{rk}(B_V) + \text{rk}(A_2) - \text{rk}(\hat{A}_2) \\ &= \text{rk}(B_V) - \text{rk}(\hat{A}_2) + \text{rk}(A_2) \\ &= \text{rk}(A_2) - 1 \end{aligned}$$

□

But as noted above $x_j^{s_j} \in B_V$ where $j \in \{n_1 - d_1 + 1, \dots, n_1\}$ and so $\text{rk}(B_V \cap A_1) = d_1$. Now

$$\text{rk}(B_V) \geq \text{rk}(A_1 \cap B_V) + \text{rk}(A_2 \cap B_V) \geq d_1 + \text{rk}(A_2) - 1 = \text{rk}(B_V)$$

and hence $\text{rk}(B_V \cap A_2) = \text{rk}(A_2) - 1$. There will be no harm in assuming that y_1, \dots, y_{n_2-1} generate $B_V \cap A_2$ as our arguments remain valid in passing to a subgroup of A_2 of finite index (Theorem 5.1, Propositions 2.1 and Corollary 5.2). By [G, Theorem 3], the crossed product

$$F(x_1, \dots, x_{n_1-d_1})[x_{n_1-d_1+1}^{s_{n_1-d_1+1}}, \dots, x_{n_1}^{s_{n_1}}, y_1, \dots, y_{n_2}]$$

has a nonzero module finite dimensional over $F(x_1, \dots, x_{n_1-d_1})$. We can thus apply Theorem 4.4 noting remark 4.5. Again we may assume $s_i = 1$ in Theorem 4.4. We thus obtain

$$(28) \quad [\bar{a}_i \bar{x}_i^{s_i}, \bar{a}_j \bar{x}_j^{s_j}] = 1 \quad \forall i, j \in \{n_1 - d_1 + 1, \dots, n_1\},$$

$$(29) \quad [\bar{a}_i \bar{x}_i^{s_i}, \bar{b}_j \bar{y}_j] = 1 \quad \forall i \in \{n_1 - d_1 + 1, \dots, n_1\}, j \in \{1, \dots, n_2 - 1\},$$

$$(30) \quad [\bar{b}_i \bar{y}_i, \bar{b}_j \bar{y}_j] = 1 \quad \forall i, j \in \{1, \dots, n_2 - 1\},$$

where $a_i, b_j \in \langle x_1, \dots, x_{n_1-d_1} \rangle$. But (34) yields:

$$(31) \quad [a_i x_i^{s_i}, \bar{b}_j] = 1 \quad \forall i \in \{n_1 - d_1 + 1, \dots, n_1\}, j \in \{1, \dots, n_2 - 1\}.$$

By definition, $d_1 = \dim(F * A_1)$ and so $\bar{b}_j = 1$ for all j , otherwise by (28) and (31), there is a subgroup $B_1 < A_1$ with $\text{rk}(B_1) \geq d_1 + 1$ so that $F * B_1$ is commutative. But this is impossible by Theorem 5.1. Hence $b_j = 1$ for all $j \in \{1, \dots, n_2 - 1\}$. It then follows from (35) that

$$[\bar{y}_i, \bar{y}_j] = 1 \quad \forall i, j \in \{1, \dots, n_2 - 1\}.$$

Hence A_2 has a subgroup B_2 with $\text{rk}(B_2) = n_2 - 1$ so that $F * B_2$ is commutative. This is contrary to the hypothesis in the theorem that $\dim(F * A_2) < n - 1$.

In view of (26), we may now assume that $\mathcal{GK}(N_1) = n_1 - d_1 + 1$. By Proposition 3.1 (and the succeeding paragraph) this implies that the ring

$$F(x_1, \dots, x_{n_1-d_1+1})[x_{n_1-d_1+2}, \dots, x_{n_1}, y_1, \dots, y_{n_2}]$$

has a nonzero module finite dimensional over the division ring $F(x_1, \dots, x_{n_1-d_1+1})$. For the sake of convinience, we may assume that

$$(32) \quad [\bar{y}_k, \bar{y}_l] = 1 \quad \forall k, l \in \{1, \dots, d_2\}$$

noting Theorem 5.1, Propositions 2.1 and Corollary 5.2. We may apply Theorem 4.4 noting Remark 4.5. We thus have the following relations.

$$(33) \quad [\bar{a}_i \bar{x}_i, \bar{a}_j \bar{x}_j] = 1 \quad \forall i, j \in \{n_1 - d_1 + 2, \dots, n_1\},$$

$$(34) \quad [\bar{a}_i \bar{x}_i, \bar{b}_j \bar{y}_j] = 1 \quad \forall i \in \{n_1 - d_1 + 2, \dots, n_1\}, j \in \{1, \dots, n_2\},$$

$$(35) \quad [\bar{b}_i \bar{y}_i, \bar{b}_j \bar{y}_j] = 1 \quad \forall i, j \in \{1, \dots, n_2\},$$

where $a_i, b_j \in \langle x_1, \dots, x_{n_1-d_1+1} \rangle$. But as \mathcal{A} is a tensor product, (32) and (35) together mean that

$$(36) \quad [\bar{b}_k, \bar{b}_l] = 1 \quad \forall k, l \in \{1, \dots, d_2\}$$

On the other hand (34) yields:

$$(37) \quad [\bar{a}_i \bar{x}_i, \bar{b}_j] = 1 \quad \forall i \in \{n_1 - d_1 + 2, \dots, n_1\}, j \in \{1, \dots, n_2\}.$$

Now the $\bar{a}_i \bar{x}_i$ are $d_1 - 1$ independent commuting monomials all of which commute with the \bar{b}_j . But as $\{\bar{b}_1, \dots, \bar{b}_{d_2}\}$ commute mutually, we must have

$$\text{rk}(\langle b_1, \dots, b_{d_2} \rangle) = 1.$$

For otherwise we would obtain a contradiction to Theorem 5.1 (applied to $F * A_1$).

As $d_2 \geq 2$ by the hypothesis in the theorem, hence b_1 and b_2 are mutually dependent. We may thus find integers t_1 and t_2 so that $b_1^{t_1} = b_2^{t_2}$. Pick j so that $d_2 + 1 \leq j \leq n_2$. We have from (35):

$$[b_1, b_j][y_1, y_j] = 1$$

and so

$$[b_1^{t_1}, b_j][y_1^{t_1}, y_j] = 1.$$

Similarly

$$[b_2^{t_2}, b_j][y_2^{t_2}, y_j] = 1.$$

whence combining the last two equations we get

$$[y_1^{t_1} y_2^{-t_2}, y_j] = 1$$

But this means that $F * A_2$ has center larger than F contrary to the hypothesis in the theorem. This concludes our proof. \square

Corollary 5.15. *Suppose that the algebras $F * A_1$ and $F * A_2$ satisfy the following conditions*

- (i) $\dim(F * A_1), \dim(F * A_2) \geq 2$,
- (ii) $\text{co-dim}(F * A_1) \geq \text{co-dim}(F * A_2) = 2$,
- (iii) $F * A_i$ has center F .

Then

$$\dim(F * A_1 \otimes F * A_2) = \dim(F * A_1) + \dim(F * A_2)$$

Proof. Let $T := F * A_1 \otimes F * A_2$, $n_i = \text{rk}(A_i)$ and $d_i = \dim(F * A_i)$, where $i \in \{1, 2\}$. Hypothesis (ii) in the theorem means that $n_1 - d_1 \geq n_2 - d_2$ and thus $n_2 + d_1 \leq n_1 + d_2$. By Theorem 5.14 we get

$$\begin{aligned} \dim(T) &< \min\{d_1 + n_2, d_2 + n_1\} - 1 \\ &= \dim(F * A_1) + \text{rk}(A_2) - 1 \\ &= \dim(F * A_1) + \dim(F * A_2) + 1. \end{aligned}$$

The corollary now follows from Corollary 5.3. \square

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